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A Note on *A*-linear Operators on Banach *A*-module

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Abstract

Let A be a Banach $f-$algebra. In this paper we are interested in $A-$linear operators on a Banach $A-$module.

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Introduction

Let A be a Banach $f-$algebra and let X be a Banach space. By $L(X)$ we denote the set of all continuous linear opeartors from X into X. The topological dual of X will be denoted by X'. We say that X is a Banach $A-\text{mod}\,ule$ if there exists a bilinear mapping

$$\begin{aligned} p : A \times X &\to X \\ (a, x) &\to a.x \end{aligned}$$ satisfying the following conditions :

$(i)\ 1.x = x \quad for \ \ all \ \ x \in X\ , 1 \in A,$

$(ii)\ (ab).x = a.(b.x) \ \ for \ \ all \ \ a,b \in A\ ,\ x \in X\ ,$

$(iii)\ \|a.x\| \leq \|a\|\|x\| \ \ for \ \ all \ \ a \in A\ ,\ x \in X\ .$

Bilinear mapping p induces $m : A \to L(X)$, $(a,x) \to a.x = m(a)x$, is a unital, norm $\|.\|$ to strong operator topology continuous, algebra homomorphism. Hence, we accomplish the following three other bilinear mappings :

$(1)\ X \times X' \to A'$

$(x, f) \to (x.f)(a) = f(a.x), \quad for \quad x \in X,\ f \in X', a \in A;$

$(2)\ A'' \times X' \to X'$

$(a, f) \to (a.f)(x) = a(x.f), \quad for \quad a \in A'',\ f \in X', a \in A;$

$(3)\ A'' \times X'' \to X''$

$(a, z) \to (a.z)(f) = z(a.f), \quad for \quad a \in A'', z \in X'',\ f \in X'.$

When X is taken as A, then (3)becomes the Arens product on A''**.** The bilinear mapping (2) defines a Banach $A-$ module structures on X' that gives a homomorphism $m^* : A'' \to L(X')$ defined by $m^*(a)f = a.f$. The bilinear mapping (3) defines a Banach $A''-$ module structure on X''. These are called the Arens extensions of the module multiplication X ,[6].

Lemma 1 : Let X be a Banach $A-$ module. The following assertions are true :

(i) For each $a \in A$,

$m^*(a) = (m(a))^*$, where $(m(a))^*$ is the adjoint of $m(a)$.

(ii) X' is a Banach $A''-$module.

(iii) X'' is a Banach $A''-$module.

(iv) For $a \in A''$, $m^*(a)$ is continuous from $X'[\sigma(X', X)]$ into $X'[\sigma(X', X)]$.

(v) m^* is continuous from $A''[\sigma(A'', A')]$ into $L(X')[w^* ot]$, where $w^* ot$ is the weak * operatör topology.

Proof:

(i) For all $a \in A$,

$m^*(a)f = a.f$

$$[m^*(a)f](x) = (a.f)(x)$$

$$= a(f.x)$$

$$= f(a.x)$$

$$= f(m(a)x)$$

$$= ((m(a))^* f)(x)$$

(ii) - $To\ show\ 1.f = f\ for\ all\ f \in X', 1 \in A''$

$(1.f)(x) = f(1.x) = f(x)$ $then,$

$$1.f = f\ .$$

- $To\ show\ (a''b'').f = a''(b''.f)\ for\ all\ a'', b'' \in A''\ and\ f \in X'$

$Take\ a'', b'' \in A''$ and $a_\alpha, a_\beta \in A$ as

$\lim a_\alpha = a''$

$\lim a_\beta = b''$.

$\lim(a_\alpha a_\beta) = a''.b''$ then ,

$(a_\alpha.a_\beta).f = a_\alpha(a_\beta.f)$

$\lim(a_\alpha.a_\beta).f = \lim a_\alpha(a_\beta.f)$ and,

$(a''.b'').f = a''.(b''.f)$.

- We claim $\|a''.f\| \le \|a''\|\|f\|$ $\quad$ *for all* $a'' \in A''$ *ve* $f \in X'$

Since mapping p is bilinear continuous, mappings (1), (2), (3) are bilinear continuous too. So, $\|a''.f\| \le \|a''\|\|f\|$.

(iii) we can prove this same with (ii).

(iv) Let we take $a_\alpha \in A$ as $a_\alpha \to a$. Then, for all $f \in A'$, to operator topology $\sigma(A'', A')$ is $a_\alpha(f) \to a(f)$.

For $m^* : A'' \to L(X')$,

When we take $a_\alpha \to a$ for x';

$a_\alpha.x' \to a.x'$ by the continuity of (3)

$(m^*(a_\alpha)x')x \to (m^*(a)x')x$ and

$m^*(a_\alpha) \to m^*(a)$. Then $m^* : A'' \to L(X')$ is continuous from $A''[\sigma(A'', A')]$ into $L(X')[w^*ot]$.

Definition 2 [1]: Let X be a Banach $A-$ module , $x \in X$. Then ,

$\Delta(x) = Cl_X\{a.x : a \in A, \|a\| \le 1\}$,

where Cl_X denotes the closure in X . Let Y be a subspace of X . Y is called an ideal if for each $x \in Y$, $\Delta(x) \in Y$. Let X be a Banach $A-$ module and let $f \in X'$, then $\Delta(f) = Cl_{X'}\{a.f \; : \; a \in A'', \|a\| \le 1\}$, where $Cl_{X'}$ denotes the closure in X'.

Definition 3 [1]: Let X be a Banach $A-$ module and $x, y \in X$. $x, y(x \perp y)$ is called disjoint if $\Delta(x+y)=\Delta(x)+\Delta(y)$ *and* $\Delta(x)\cap\Delta(y)=\{0\}$. Let X be a Banach A_1 $-$module and let Y be a Banach A_2 $-$module and suppose that $T: X \to Y$ is a linear operator. Then T is called disjointness preserving operator (d-homomorphism), if $x \perp z$ implies $Tx \perp Tz$.

Let X, Y be two Banach $A-$modules (with the same A). A linear continuous operator $T: X \to Y$ is called $A-$ linear (or a $A-orthomorphism$) if $T(a.x)=a.Tx$, *for all* $a \in A$, $x \in X$. Take $A=C(K)$. Then, if a linear operator $T: X \to Y$ is an $A-$ linear, then adjoint T' of T is disjointness preserving operator from Y' *in to* X' ,[4].

Theorem 4 : Let X ,Y be two Banach $A-$ modules. If a linear operator $T: X \to Y$ is an $A-$ linear, then its continuous adjoint operator $T': Y' \to X'$ is an $A''-$ linear operator.

Proof : Firstly , let us satisfy that $T'(a.y')=a.T'y'$ for $a \in A$, $y' \in Y'$. Take an arbitrary $x \in X$, we show that $T'(a.y')(x)=(a.T'y')(x)$.

$T'(a.y')(x)=(a.y')(Tx)=y'(a.Tx)$ (by the bilinear mapping (1))

$= y'(T(a.x))$ (by the $A-$*linearity*)

$T'y'(a.x)=a(x.T'y')$ (by the bilinear mapping (1))

$=(a.T'y')(x)$ (by the bilinear mapping (2)).

Hence, T' is a $A-$ linear. It is well-known that A is $|\sigma|(A'', A')-$dense in A''[2]. Let $a \in A''$. There exists a net $\{a_\alpha\}$ *in* A such that $a_\alpha \to a$ in $\sigma(A'', A')$. By the continuity of the bilinear mapping (2), we get $a_\alpha.y' \to a.y'$ for $a.y' \in Y'$. Since T' is continuous, it follows that $T'(a_\alpha.y') \to T'(a.y')$. By the first case, we have $T'(a_\alpha.y')=a_\alpha.T'y'$. By the bilinear mapping (2), $a_\alpha.T'y' \to a.T'y'$ and hence $T'(a.y')=a.T'y'$. Therefore, $T'$is $A''-$linear.

Proposition 5 : Let X be a Banach $A-$module and let $T: X \to X$ be a $A-$linear operator. Then,

$Tx.x' = x.T'x'$ for all $x \in X$, $x' \in X'$, where $T' : X' \to X'$ is continuous adjoint of T.
We can prove this proposition by using the bilinear mappings (2) and p.

Let X be a Banach $A-$module. Then , we define the set $Orth_A(X)$ as the set of all $A-$linear mappings.

Corollary 6 : If $T \in Orth_A(X)$, then $T' \in Orth_{A''}(X')$.

Suppose that X is a Banach $A-$module. Then we define the set $W(X')$ as the set of all continuous linear operators $T' : X' \to X'$ such that $TE \subseteq E$ for each w^*- closed ideal E of X' . In particular, we get the following corollary:

Corollary 7 [5]: Let X be a Banach $C(K)-$ module. Then , $W(X') = Orth_{C(K)''}(X')$.

Proof : Assume that $T \in W(X')$. Then, it is easy to see that T is a $C(K)-$linear. Using that $C(K)$ is $|\sigma|\left(C(K)'', C(K)'\right)$ dense in $C(K)''$ and continuity of T, we see that $W(X') \subseteq Orth_{C(K)''}(X')$. For the converse inclusion, we use Lemma 9.7 of [1], or Theorem 2 of [3].

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